SELECTION PRINCIPLES IN UNIFORM SPACES AND MAPPINGS

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Introduction

The theory of the selection principles is one of the main directions of set-theoretic topology and at the moment it is quite extensive and more applications in various areas of mathematics. At present, the theory of the selection principles is historically, logiccaly justified and has been far advanced thanks to these [2]-[35] and other fundamentals works.
In connection with the problem posed by Z. Frolik about the "uniformization" of topological properties, Lj. Kočinac [7] found uniform analogues of the most important properties of the selection principles [6], [11], [15] : uniformly Menger, uniformly Hurewicz, uniformly Rothberger spaces and e.t.c.

These properties are considered as types of totally bounded uniform spaces, for example, the uniformly Menger space occupies an intermediate place between precompact and pre-Lindelöf spaces, but it does not have the properties that precompact and pre-Lindelöf spaces had.

The work consists of two parts, in the first part investigates of uniformly Menger, uniformly Hurewicz and uniformly Rothberger spaces. The most important properties and of these classes of uniform spaces are established. In the second part, Menger mappings of topological spaces are introduced and investigated.

Preliminaries and Denotations

Throughout this paper all uniform spaces are assumed to be Hausdorff and mappings are uniformly continuous.
For covers α and β of a set X, we have :
$\alpha \wedge \beta = \{A \bigcap B | A \in \alpha, B \in \beta\}$, $\alpha \wedge \{X\} = \{A \bigcap X | A \in \alpha\}$.
$\alpha(x) = \bigcup St(\alpha, x)$, $St(\alpha, x) = \{A \in \alpha | A \ni x\}$, $x \in X$,
$\alpha(H) = \bigcup St(\alpha, H)$, $St(\alpha, H) = \{A \in \alpha | A \bigcap H \neq \varnothing\}$, $H \subset X$.
For covers α and β of the set X, the symbol $\alpha \prec \beta$ means that the cover α is a refinement of the cover β, i.e. for any $A \in \alpha$ there exists $B \in \beta$ such that $A \subset B$ and, the symbol $\alpha \prec *\beta$ means that the cover α is a strongly star refinement of the cover β, i.e. for any $A \in \alpha$ there exists $B \in \beta$ such that $\alpha(A) \subset B$.

A uniformity on a nonempty set X is a family U of covers of X which satisfies the following conditions :

1. If $\alpha \in U$ and if β is a cover of X such that $\alpha \prec \beta$, then $\beta \in U$.
2. If $\alpha_1, \alpha_2 \in U$, then there exists $\alpha \in U$ such that $\alpha \prec \alpha_1$ and $\alpha \prec \alpha_2$.
3. If $\alpha \in U$, then there exists $\beta \in U$ such that $\beta \prec *\alpha$.
4. For any two distinct points $x, y \in X$ there is an $\alpha \in U$ such that no member of α contains both x and y.

The covers from U are called uniform covers, and the pair (X, U) is called a uniform space.
The topology on X generated by U is denoted by $\tau_U = \{O \subset X : \forall x \in O \exists \alpha \in U : \alpha(x) \subset O\}$.

If U and V are two uniformities on a set X and $U \subset V$, then we say that the uniformity V is finer than the uniformity U. The finest uniformity on the Tychonoff space X is called the universal uniformity on the space X and is denoted by U_X.
Let $\exp X$ be the set of all non-empty closed subsets of the space (X, τ_U). For each $\alpha \in U$ we put $P(\alpha) = \{\langle\alpha'\rangle : \alpha' \subset \alpha\}$, where $\langle\alpha'\rangle = \{F \in \exp X : F \subset \bigcup \alpha', F \bigcap A \neq \varnothing \forall A \in \alpha'\}$. A uniform space $(\exp X, \exp U)$ is called the hyperspace of closed subsets of a uniform space (X, U) and the uniformity $\exp U$ is called the Hausdorff uniformity on $\exp X$.

A uniform space (X, U) is called :

1. *precompact*, if the uniformity U has a base consisting of finite covers [1];

2. *σ-precompact (σ-compact)*, if it can be represented as the union of countably many precompact (compact) subspaces [1];

3. *totally bounded*, if each $\alpha \in U$ has a finite set $H \subset X$ such that $\alpha(H) = X$ [1];

4. *pre-Lindelöfor or $\aleph_0$-bounded*, if the uniformity U has a base consisting of countable cover [1], [7], [9];

5. *uniformly locally precompact*, if the uniformity of U contains a uniform cover consisting is precompact sets [1];

6. *uniformly Menger space*, or has the *uniform Menger property*, if for each sequence $(\alpha_n | n \in N) \subset U$ there is a sequence $(\beta_n | n \in N)$ such that for each $n \in N$, β_n is a finite subset of α_n and $\bigcup_{n \in N} \beta_n$ is a cover of X [7];

7. *uniformly Hurewicz space* or has the *uniform Hurewicz property* if for each sequence $(\alpha_n | n \in N) \subset U$ there is a sequence $(\beta_n | n \in N)$ such that each β_n is a finite subset of α_n and for each $x \in X$ we have $x \in \bigcup \beta_n$ for all but finitely many n [7];

8. *uniformly Rothberger space* or has the *uniform Rothberger property* if for each sequence $(\alpha_n | n \in N) \subset U$ there is a sequence $(A_n | n \in N)$ such that for each $n \in N$ $A_n \in \alpha_n$ and $\bigcup_{n \in N} A_n = X$ [7].

Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping of a uniform space (X, U) onto a uniform space (Y, V). The mapping f is called :

1. *precompact*, if for each $\alpha \in U$ there exist a uniform cover $\beta \in V$ and a finite uniform cover $\gamma \in U$ such that $f^{-1}\beta \wedge \gamma \prec \alpha$ [1];
2. *uniformly perfect*, if it is both precompact and perfect [1];
3. *twice uniformly continuous*, if for each $\alpha \in U$ there exist a uniform cover $\beta \in V$ such that $f^{-1}\beta \prec \alpha^{\angle}$, $\alpha^{\angle} = \{\bigcup \alpha_0 : \alpha_0 \subset \alpha - \textit{finite}\}$ [13].

A cover α of uniform space (X, U) is called *co-cover*, if $\alpha \bigcap F \neq \varnothing$ for all free Cauchy filters F of (X, U). For the uniformity U by τ_U we denote the topology generated by the uniformity.

Uniformly Menger, uniformly Hurewicz and uniformly Rothberger properties

Let (X, U) be a uniform space.
Theorem 3.1. Any σ-precompact uniform space (X, U) is a uniformly Hurewicz space.
Proof. Let (X, U) be a σ-precompact space and $(\alpha_n | \, n \in N) \subset U$ be an arbitrary sequence. Then for any $n \in N$ the cover $\alpha_{X_n} = \alpha_n \wedge \{X_n\}$ of X_n contains a finite subcover $\alpha^0_{X_n} \subset \alpha_{X_n}$. Put $(\beta_{X_n} | \, n \in N)$, $\beta_{X_n} = \alpha^0_{X_n}$. Then for each $n \in N$, $(\beta_{X_n} | \, n \in N)$ is a finite subset of α_n and for each $x \in X$ we have $x \in \bigcup \beta_{X_n}$ for all but finitely many n. Hence (X, U) is uniformly Hurewicz space.
Corollary 3.1. Any σ-compact uniform space (X, U) is a uniformly Hurewicz space.

Corollary 3.2. Any σ-compact Tychonoff space X is a Hurewicz space.
Corollary 3.3. Any σ-precompact uniform space (X, U) is a uniformly Menger space.
Corollary 3.4. Any precompact uniform space (X, U) is a uniformly Hurewicz space.
Theorem 3.2. Any countable descrete uniform space (X, U) is a uniformly Rothberger space.
Proof. Let (X, U) be a countable discrete space and $(\alpha_n | \, n \in N) \subset U$ be a sequence of uniform covers. Since the uniform space (X, U) is a countable discrete space, there exists a base $B = \{\alpha\}$ consisting of a countable covering $\alpha = \{\{x_n\} : \; x_n \in X\}$. It's clear that $\alpha \prec \alpha_n$ for all $n \in N$. That $(A_n | \, n \in N)$ is the desired sequence.
Proposition 3.1. The space of real numbers R with natural uniformity U_R is a uniformly Hurewicz space,

but is not precompact.

Proof. Let $(\alpha_n | \ n \in N) \subset U_R$ be an arbitrary sequence of uniform covers and $\lambda = ((-n, n) | \ n \in N)$ be an open cover of the space (R, U_R). Consider the following construction : for $n = 1$ due to the compactness $[-1, 1]$ from the cover α_1 select a finite subfamily $\alpha_1^0 \subset \alpha_1$ such that $(-1, 1) \subset [-1, 1] \subset \bigcup \alpha_1^0$, for $n = 2$ from the cover α_2 select a finite subfamily $\alpha_2^0 \subset \alpha_2$ such that $(-2, 2) \subset [-2, 2] \subset \alpha_2^0$, and for $n = 3$ from the cover α_3 select a finite subfamily $\alpha_3^0 \subset \alpha_3$, such that $(-3, 3) \subset [-3, 3] \subset \alpha_3^0$, etc. Continuing this process, get a sequence $(\alpha_n^0 | \ n \in N)$ of finite subfamilies. Since λ is a cover of the space (R, U_R) and each element $(-n, n) \in \lambda$ is covered by some finite subfamily α_n^0, then for each $x \in X$ we have $x \in \bigcup \alpha_n^0$ for all but finitely many n. Therefore, the space (R, U_R) is uniformly Hurewicz space.

Corollary 3.5. The space of real numbers R with natural uniformity U_R is a uniformly Menger space.
Corollary 3.6. The space of real numbers R with topology τ_{U_R} induced by natural uniformity U_R is a Menger space, but is not compact.
Proposition 3.2. The space of natural numbers N with discrete uniformity U_D is a uniformly Rothberger space.
Proof. Let $(\alpha_n | \, n \in N) \subset U_D$ be an arbitrary sequence of uniform covers and $B = \{\beta\}$, $\beta = (\{n\} | \, n \in N)$ is her base. It is clear that $\beta \prec \alpha_n$ for each $n \in N$. For each $n \in N$ we choose $A_{\alpha_n} \in \alpha_n$, such that $\{n\} \subset A_{\alpha_n}$. It is easy to see that the family $(A_{\alpha_n} | \, n \in N)$ forms a cover of the discrete uniform space (N, U_D).

Corollary 3.7. The space of natural numbers N with topology τ_{U_D} induced by discrete uniformity U_D is a Rothberger space.
Theorem 3.3. For a uniformly locally precompact space (X, U) the following conditions are equivalent :
1. (X, U)-uniformly Menger ;
2. (X, U)-pre-Lindelöf.
Proof. 1. $\Rightarrow$ 2. Let now $\alpha \in U$ be an arbitrary uniform cover of a uniformly Menger space (X, U). Put $\alpha_n = \alpha$ for any $n \in N$. Then for the sequence $(\alpha_n | \, n \in N) \subset U$, where $\alpha_n = \alpha$, $n \in N$, there is a sequence $(\beta_n | \, n \in N) \subset U$ on finite subfamilies such

that for any $n \in N$, β_n is a subfamily of the cover α_n, i.e. α and $\bigcup_{n \in N} \beta_n$ is a cover of the space (X, U). For each $n \in N$ and for each element $B_{\beta_{n(i)}} \in \beta_n$, $i = 1, 2, ..., k$ by selecting one element $A_{B_{\beta_{n(i)}}}$ from $\alpha = \alpha_n$, we obtain a finite subfamily $\alpha_n^0 \subset \alpha$. Then $\bigcup_{n \in N} \alpha_n^0$ is a countable subfamily of the cover α. Since the family $\bigcup_{n \in N} \beta_n$ is a cover of the space (X, U), the family $\bigcup_{n \in N} \alpha_n^0$ is also a cover of (X, U). Therefore, the space (X, U) is a pre-Lindelöf space.

2. $\Rightarrow$ 1. Let (X, U) be a pre-Lindelöf space. Let's show that (X, U) is a uniformly Menger space. Let $(\alpha_n | \, n \in N) \subset U$ be an arbitrary sequence and β a countable uniform cover consisting of precompact subsets, i.e. $\beta = \{B_1, B_2, ...B_n, ...\}$. For any $n \in N$ due to the precompactness of B_n, select from the cover α_n a finite subfamily $\alpha_n^0 \subset \alpha$ such that $B_n \subset \bigcup \alpha_n^0$.

Since β is a cover of the space, the family $\bigcup_{n \in N} \alpha_n^0$ is also a cover of the space (X, U). Consequently, (X, U) is uniformly a Menger space.

Theorem 3.4 The product $(X \times Y, U \times V)$ of a uniformly Menger space (X, U) and σ-precompact uniform space (Y, V) is a uniformly Menger space.

Proof. Let $(X \times Y, U \times V)$ be the product of the uniformly Menger space (X, U) and the σ-precompact space (Y, V), and let $(\gamma_{\ n}| \ n \in N) \subset U \times V$ be an arbitrary sequence of uniform covers. Let $\gamma_n = \alpha_n \times \beta_n$, $\alpha_n \in U$, $\beta_n \in V$ for any $n \in N$. Since the space (X, U) is uniformly Menger there exists a sequence $(\sigma_n| \ n \in N)$ such that for any $n \in N$, $(\sigma_n| \ n \in N)$ is a finite subfamily of α_n and $\bigcup\limits_{n \in N} \sigma_n$ is a cover of the space (X, U), and for any $n \in N$ the cover $\beta_{X_n} = \beta_n \wedge \{X_n\}$ of X_n contains a finite subcover $\beta^0_{X_n} \subset \beta_{X_n}$. Then exist $\beta^{X_n}_n = \{B^{X_n}_j : j = 1, 2, ..., k\}$, $B^{X_n}_j \in \beta_n$ such that $\beta^{X_n}_n \wedge X_n = \beta^0_{X_n}$. Put $\beta^0_n = (\beta^{X_n}_n \big| \, n \in N)$. Then the family $(\sigma_{\ n} \times \beta^0_n \big| \, n \in N)$ is a finite subfamily of the cover $\alpha_n \times \beta_n$. Let's show that the family $\bigcup\limits_{n \in N} \sigma_n \times \beta^0_n$ is a cover of the space $(X \times Y, U \times V)$. Let $(x, y) \in X \times Y$ be an arbitrary point. Then there are a number "n^*" and $A \in \sigma_{n^*}$ such that $x \in A$. Since β^0_n is a finite subcover of the space (Y, V), then for any $n \in N$, and even more so for n^*, there exists $B \in \beta^0_{n^*}$ such that $y \in B$. Hence $(x, y) \in A \times B \in \sigma_{n^*} \times \beta^0_{n^*}$. Therefore, the space $(X \times Y, U \times V)$ is uniformly a Menger space.

Theorem 3.5. The finite discrete sum $(X, U) = \coprod\{(X_i, U_i) : i = 1, 2, ..., m\}$ uniformly Menger (uniformly Hurewicz, uniformly Rothberger) spaces $(X_i, U_i), i = 1, 2, ..., m$ is uniformly Menger (uniformly Hurewicz, uniformly Rothberger).
Proof. The proof of the theorem proceeds similarly in all three cases, so we consider only the uniformly Menger case. Let $(X_i, U_i), i = 1, 2, ..., m$ be a uniformly Menger space. Let $(\alpha_n| \, n \in N) \subset U$ be an arbitrary sequence. Then $(\alpha_{\,n,X_i}\big| \, n \in N) \subset U_i$, $\alpha_{n,X_i} = \alpha_n \bigcap X_i$, $i = 1, 2, ..., m$. Therefore, for every $i \in \{1, 2, ..., m\}$ there is a sequence $(\beta_{n,X_i}\big| \, n \in N)$ such that for any $n \in N$ the family β_{n,X_i} is finite and $\bigcup\limits_{n \in N} \beta_{n,X_i}$ is a cover of the space (X_i, U_i). Put $\beta_n = \bigcup\limits_{i=1}^{m} \beta_{n,X_i}$.

Since the space $(X_i, U_i), i = 1, 2, ..., m$ are pairwise disjoint in the space (X, U), then β_n is a finite subfamily for α_n. According to the definition of a discrete sum of uniform spaces, we have the family $\bigcup_{n \in N} \beta_n$ is a cover of the space (X, U). Therefore, the uniform space (X, U) is uniformly Menger.

Proposition 3.3. The completion of a uniformly Menger (uniformly Hurewicz, uniformly Rothberger) space is a uniformly Menger (uniformly Hurewicz, uniformly Rothberger) space.

Proof. Let's consider only the uniformly Rothberger case and the remaining cases proceed similarly. Let $(\tilde{X}, \tilde{U})$ be the completion of the uniformly Rothberger space (X, U) and $(\tilde{\alpha}_n | n \in N) \subset \tilde{U}$ be an arbitrary sequence. Put $\alpha_n = \tilde{\alpha}_n \wedge \{X\}$. Then from the construction ([1]) of completion of uniform spaces $(\alpha_n | n \in N) \subset U$. Since (X, U) is a uniformly Rothberger space, there exists a sequence $(A_n | n \in N)$ such that for any $n \in N$, $A_n \in \alpha_n$ and $\bigcup\limits_{n \in N} A_n$ is a cover of the space (X, U). It is easy to see that $\bigcup\limits_{n \in N} \tilde{A}_n$ is a cover of the $(\tilde{X}, \tilde{U})$. Consequently, $(\tilde{X}, \tilde{U})$ is a uniformly Rothberger space.

Theorem 3.6. The remainder $(\tilde{X} \backslash X, \tilde{U}_{\tilde{X} \backslash X})$ of a uniform space (X, U) is a uniformly Rothberger space if and only if for any sequence $(\alpha_n | \, n \in N) \subset U$ there exists a sequence $(A_n | \, n \in N)$ such that $\bigcup\limits_{n \in N} A_n$ is a co-cover of the uniform space (X, U).

Proof. *Necessity*. Let the remainder $(\tilde{X} \backslash X, \tilde{U}_{\tilde{X} \backslash X})$ of a uniform space (X, U) be a uniformly Rothberger space and $(\alpha_n | \, n \in N) \subset U$ be an arbitrary sequence of uniform covers of the space (X, U). Then $(\hat{\alpha}_n | \, n \in N) \subset \tilde{U}_{\tilde{X} \backslash X}$, where

$\hat{\alpha}_n = \tilde{\alpha}_n \bigcap (\tilde{X} \backslash X)$, $\tilde{\alpha}_n = \{\tilde{A}_n : A_n \in \alpha\}$, $\tilde{A}_n = \tilde{X} \backslash [X \backslash A_n]_{\tilde{X}}$. Since $(\tilde{X} \backslash X, \tilde{U}_{\tilde{X} \backslash X})$ is uniformly Rothberger space, there exists a sequence $(\hat{A}_n \Big| n \in N)$, $\hat{A}_n \in \hat{\alpha}_n$ such that $\bigcup\limits_{n \in N} \hat{A}_n$ is a cover of the remainder $(\tilde{X} \backslash X, \tilde{U}_{\tilde{X} \backslash X})$. Put $A_n = \tilde{A}_n \bigcap X$. Let F be an arbitrary free Cauchy filter of the space (X, U). Then it conver to some point $\hat{x} \in \tilde{X} \backslash X$. There is $\hat{A} \in \bigcup\limits_{n \in N} \hat{A}_n$ such that $\hat{A} \ni \hat{x}$, $\hat{A} = \tilde{A} \bigcap (\tilde{X} \backslash X)$. Denote by $\tilde{B}(\hat{x})$ the filter of the neighborhoods of the point $\hat{x}$. Note $\tilde{B}(\hat{x}) \bigcap X = F$. Then $A \in F$. Therefore, $\bigcup\limits_{n \in N} A_n \bigcap F \neq \varnothing$, i.e. the family $\bigcup\limits_{n \in N} A_n$ is a co-cover of the space (X, U) .

Sufficiency. Let $(\hat{\alpha}_n | \, n \in N)$ be an arbitrary sequence of uniform covers of the space $(\tilde{X} \backslash X, \tilde{U}_{\tilde{X} \backslash X})$. Then there is a sequence $(\alpha_n | \, n \in N)$ of uniform covers of the space (X, U) such that

$(\tilde{\alpha}_n \bigcap (\tilde{X} \backslash X) \Big| n \in N) = (\hat{\alpha}_n | n \in N)$. Hence by the conditions of the theorem that there exists a sequence $(A_n | n \in N)$ such that the family $\bigcup\limits_{n \in N} A_n$ is a co-cover of the space (X, U). Put $\bigcup\limits_{n \in N} \hat{A}_n$, where $\hat{A}_n = \tilde{A}_n \bigcap (\tilde{X} \backslash X)$, $\tilde{A}_n = \tilde{X} \backslash [X \backslash A_n]_{\tilde{X}}$. Let show that the family $\bigcup\limits_{n \in N} \hat{A}_n$ is a cover of the space $(\tilde{X} \backslash X, \tilde{U}_{\tilde{X} \backslash X})$. Let $\hat{x} \in \tilde{X} \backslash X$ be an arbitrary point. Denote by $\tilde{B}(\hat{x})$ the filter of neighborhoods of the point $\hat{x} \in \hat{X}$. Put $F = \tilde{B}(\hat{x}) \bigcap X$. Then it is easy to see that F is a free Cauchy filter of the space (X, U) . From this it follows that $\bigcup\limits_{n \in N} A_n \bigcap F \neq \varnothing$, i.e. there exists $A \in \bigcup\limits_{n \in N} A_n$ such that $A \in F$. It's clear that $F \in \tilde{A} \in \tilde{B}(\hat{x})$. Hence, $\hat{A} \ni \hat{x}$, $\hat{A} \in \bigcup\limits_{n \in N} \hat{A}_n$, $\hat{A} = \tilde{A} \bigcap (\tilde{X} \backslash X)$. Consequently, the family $(\hat{A}_n \Big| n \in N)$ is a cover of the remainder $(\tilde{X} \backslash X, \tilde{U}_{\tilde{X} \backslash X})$.

Thus, the remainder $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ is uniformly Rothberger space.

Theorem 3.7. The remainder $(\tilde{X}\backslash X, \tilde{U}_{\tilde{X}\backslash X})$ of a uniform space (X, U) is uniformly Hurewicz space if and only if for any sequence $(\alpha_n | n \in N) \subset U$ there exists a sequence of $(\alpha_n^0 | n \in N)$ finite subfamilies and for each free Cauchy filter F of (X, U) we have $\bigcup \alpha_n^0 \in F$ for all but finitely many n.

The proof Theorem with minor modifications is similar to the proof of Theorem 3.6.

Theorem 3.8. For a uniform space (X, U) the following conditions are equivalent :
1. (X, U)-uniformly Rothberger ;
2. $(\exp X, \exp U)$-uniformly Rothberger.
Proof. Prove the implications 1. $\Rightarrow$ 2. Let (X, U) be a uniformly Rothberger space and $(P(\alpha_n)| n \in N)$, where
$P(\alpha_n) = \{\langle\alpha'_n\rangle : \alpha'_n \subset \alpha_n\}$,
$\langle\alpha'_n\rangle = \{F \in \exp X : F \subseteq \bigcup \alpha'_n, F \bigcap A_n \neq \varnothing, A_n \in \alpha'_n\}$ sequence of covers of hyperspace $(\exp X, \exp U)$. Then for a sequence $(\alpha_n| n \in N)$ of uniform covers of the space (X, U) there is a sequence $(A_n| n \in N)$ such that $A_n \in \alpha_n$ and family $(A_n| n \in N)$ is a cover of the space (X, U). Let us show that the family $\{\langle\alpha'_n\rangle\}$, $\langle\alpha'_n\rangle \in P(\alpha_n)$ is a cover of the hyperspace $(\exp X, \exp U)$. Let $F \in \exp X$ be an arbitrary element.

Since $\{A_n\}$ is a cover of the space (X, U), then there exists $k \leq n$ such that $A_k \bigcap F \neq \varnothing$. There exist $\alpha'_k \subset \alpha_k$ such that $F \subseteq \bigcup \alpha'_k$ and $F \bigcap A \neq \varnothing$ for all $A \in \alpha'_n$, $A_k \in \alpha'_k$. Then $F \in \langle \alpha'_k \rangle$. Therefore, $\exp X \subset \bigcup\limits_{n \in N} \langle \alpha'_n \rangle$. So $(\exp X, \exp U)$ is uniformly Rothberger.

The implication 1. $\Rightarrow$ 2. follows from the fact that the a subspace of the uniformly Rothberger space is uniformly Rothberger.

Theorem 3.9. Let $f : (X, U) \to (Y, V)$ be a twice uniformly continuous mapping of a uniform space (X, U) onto a uniform space (Y, V). If one of the uniform spaces (X, U) and (Y, V) is uniformly Menger, then another uniform space is also uniformly Menger space.

Proof. Necessity follows from Kočinac's Theorem 6 (see [7], p. 134).

Sufficiency. Let $f : (X, U) \to (Y, V)$ be a twice mapping of a uniform space (X, U) onto a uniformly Menger space (Y, V) and $(\alpha_n | \, n \in N) \subset U$ be an arbitrary sequence of uniform covers. Then for any $n \in N$ there exists a cover $\beta_n \in V$, such that $f^{-1}\beta_n \prec \alpha_n^{\angle}$. Since (Y, V) is a uniformly Menger space, then for the sequence $(\beta_n | \, n \in N) \subset V$ there exists a sequence $(\beta_n^o | \, n \in N)$ of finite subfamilies such that $\bigcup\limits_{n \in N} \beta_n^o$ is a cover of the space (Y, V). Then for any $B_{n,i}^o \in \beta_o$ choose $\bigcup\limits_{j=1}^{l} A_{n,i}^{0,j} \in \alpha_n^{\angle}$, such that $f^{-1}B_{n,i}^0 \subset \bigcup\limits_{j=1}^{l} A_{n,i}^{0,j}$.

Put $\alpha_n^0 = \{A_{n,1}^{0,j}, A_{n,2}^{0,j}, ..., A_{n,k}^{0,j} : j = 1, 2, ..., l\}$. It is easy to see that the family $\bigcup\limits_{n \in N} \alpha_n^o$ is a cover of the space (X, U). Therefore, (X, U) is a uniformly Menger space.

Theorem 3.10. If a uniform space (Y, V) is a uniformly continuous image of a uniformly Rothberger space (X, U), then (Y, V) is also uniformly Rothberger.

Proof. Let $f : (X, U) \to (Y, V)$ be a uniformly continuous mapping of a uniformly Rothberger space (X, U) onto a uniform space (Y, V). Let $(\beta_n | \, n \in N)$ be a sequence of uniform covers of (Y, V). Then $(\alpha_n | \, n \in N)$ be a sequence of uniform covers of (X, U), where $\alpha_n = f^{-1}\beta_n$. Since (X, U) is uniformly Rothberger, there exists a sequence $(A_n | \, n \in N)$ such that for each $n \in N$ $A_n \in \alpha_n$ and $\bigcup\limits_{n \in N} A_n = X$, $A_n = f^{-1}B_n$.

It is easy to see that $f(\bigcup_{n\in N} A_n) = \bigcup_{n\in N} fA_n = \bigcup_{n\in N} B_n = f(X) = Y$. Then for each $n \in N$ $B_n \in \beta_n$ and $\bigcup_{n\in N} B_n = Y$. Thus, a uniform space (Y, V) is uniformly Rothberger.

A uniformly continuous mapping $f : (X, U) \to (Y, V)$ of a uniform space (X, U) onto a uniform space (Y, V) is called strong twice uniformly continuous if for any $\alpha \in U$ there exists $\beta \in V$ such that $f^{-1}\beta \prec \alpha$.

Theorem 3.11. Let $f : (X, U) \to (Y, V)$ be strong twice uniformly continuous mapping. If one of the uniform spaces (X, U) and (Y, V) is uniformly Rothberger, then another uniform space is also uniformly Rothberger.

Proof. Necessity follows from Theorem 3.10.
Sufficiency. Let $f : (X, U) \to (Y, V)$ be a twice mapping of a uniform space (X, U) onto a uniformly Rothberger space (Y, V) and $(\alpha_n | n \in N) \subset U$ - sequence of uniform covers. Then for any $n \in N$ there exists a cover $\beta_n \in V$, such that $f^{-1}\beta_n \prec \alpha_n$. Since (Y, V) is a uniformly Rothberger space, then for the sequence $(\beta_n | n \in N) \subset V$ there exists a sequence $(B_n | n \in N)$, $B_n \in \beta_n$ such that $\bigcup\limits_{n \in N} B_n$ is a cover of the space (Y, V). Then for any $n \in N$ choose $A_{B_n} \in \alpha_n$, such that $f^{-1}B_n \subset A_{B_n}$. Put $(A_{B_n} | n \in N)$. From the inclusion $X = f^{-1}Y = \bigcup\limits_{n \in N} f^{-1}B_n \subset \bigcup\limits_{n \in N} A_{B_n}$ it follows that $\bigcup\limits_{n \in N} A_{B_n}$ is a cover of the space (X, U). Therefore, (X, U) is a uniformly Rothberger space.

Menger mappings

Let $f : X \to Y$ be a continuous mapping between topological spaces X and Y.
Definition 4.1. A mapping f is called a Menger mapping if the mapping f is closed and the preimage $f^{-1}y$ of each point $y \in Y$ is a Menger space.
If f is a Menger mapping and $Y = \{y\}$, then X is a Menger space.
Recall that the mapping f is called a perfect (quasi-perfect) mapping if f is closed and the preimage $f^{-1}y$ of the $y \in Y$ is compact (Lindelöf). It is clear that every perfect mapping is a Menger mapping, and the latter is a quasi-perfect mapping, i.e.

the class of all Menger mappings is strictly between the classes of perfect and quasi-perfect mappings.

Lemma 4.1. For a mapping f the following conditions are equivalent :

1. f is a Menger mapping.
2. For each point $y \in Y$ and any sequence of covers $(\alpha_n | \, n \in \mathbb{N})$ of $f^{-1}y$ by open sets in X, there exist finite subfamilies $(\beta_n | \, n \in \mathbb{N})$ and a neighborhood O_y of the point y such that $\beta_n \subset \alpha_n$ for each $n \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} \beta_n = f^{-1}O_y$

Theorem 4.1. Let $f : X \to Y$ be a continuous mapping between spaces X and Y. If f and Y is Menger, then X is also Menger, conversely, if X is a Menger, then f is Menger.

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Thanks for attention !