Selection Principles in Topological Groups

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Introduction

Recently the theory of selection principles has been intensively developing in topological spaces, in topological groups, in uniform spaces etc.

Significant contributions to the theory of selection principles of choice were made in [1, 3, 6–13] and others.

A good overview of the very extensive literature on selection principles is contained in [3, 10].

Lj.D.R. Kočinac [3] found uniform analogues of the most important properties of the selection principles: uniformly Menger spaces, uniformly Hurewicz spaces, uniformly Rothberger space etc. These properties are considered as types of totally bounded uniform spaces, for example, the uniformly Menger space occupies an intermediate place between totally bounded and ω-bounded spaces. To each selection property of a uniform structure defined above can correspond game.

In this work we study important properties of the bounded, M-bounded, H-bounded and R-bounded topological groups.

Introduction

For covers α *and* β *of a set* X*, we have:* $\alpha \wedge \beta = \{A \cap B : A \in , B \in \beta\}$. $\alpha(x) = \cup St(\alpha, x)$, $St(\alpha, x) = \{A \in \alpha : A \ni x\}$, $x \in X$, $\alpha(H) = \cup St(\alpha, H)$, $St(\alpha, H) = \{A \in \alpha : A \cap H \neq \varnothing\}$, $H \subset X$. *For covers* α *and* β *of a set* X*, the symbol* $\alpha \succ \beta$ *means that the cover* α *is a refinement of the cover* β*, i.e. for any* $A \in \alpha$ *there exists* $B \in \beta$ *such that* $A \subset B$ *and, the symbol* $\alpha^* \succ \beta$ *means that the cover* α *is a strongly star refinement of the cover* β*, i.e. for any* $A \in \alpha$ *there exists* $B \in \beta$ *such that* $\alpha(A) \subset B$.

Introduction

Recall that a uniform space (X, U) *is called:*

(a) totally bounded, if each $\alpha \in U$ *has a finite set* $H \subset X$ *such that* $\alpha(H) = X$ *[2];*

(b) precompact, if the uniformity U *has a base consisting of finite covers [2];*

(c) ω*-bounded, if the uniformity* U *has a base consisting of countable cover [2], [3];*

(d) uniformly locally compact, if the uniformity of U *contains a uniform cover consisting is compact sets [2];*

(e) has the uniform Menger property, if for each sequence $(\alpha_n | n \in N) \subset U$ *there is a sequence* $(\beta_n | n \in N)$ *such that for each* $n \in N$, β_n *is a finite subset of* α_n *and* $\bigcup_{n \in N} \beta_n$ *is a cover of* X *[3];*

(f) has the uniform Hurewicz property if for each sequence $(\alpha_n | n \in N) \subset U$ *there is a sequence* $(\beta_n | n \in N)$ *such that each* β_n *is a finite subset of* α_n *and for each* $x \in X$ *we have* $x \in \cup \beta_n$ *for all but finitely many* n *[3];*

(g) has the uniform Rothberger property if for each sequence $(\alpha_n | n \in N) \subset U$ *there is a sequence* $(A_n | n \in N)$ *such that for each* $n \in N$ $A_n \in \alpha_n$ *and* $\bigcup_{n \in N} A_n = X$ *[3].*

Let $f : (X, U) \to (Y, V)$ *be a uniformly continuous mapping of a uniform space* (X, U) *onto a uniform space* (Y, V)*. The mapping f is called precompact, if for each* $\alpha \in U$ *there exist a uniform cover* $\beta \in V$ *and a finite uniform cover* $\gamma \in U$ *such that* $f^{-1}\beta \wedge \gamma \succ \alpha$ *[2].*

For the uniformity U *by* τ_U *we denote the topology generated by the uniformity.*

Let G be topological group and $B(e)$ be the filter of neighborhoods of the neutral element $e \in G$. Each element $W \in B(e)$ determines three covers of the group G as follows: $\alpha_l(W) = \{x \cdot W : x \in G\}$, $\alpha_r(W) = \{W \cdot x : x \in G\}$ and $\alpha_t(W) = \{x \cdot W \cdot y : x, y \in G\}$. The family of covers $\{\alpha_l(W) : W \in B(e)\}$, $\{\alpha_l(W) : W \in B(e)\}$ and $\{\alpha_l(W) : W \in B(e)\}$ forms a base of left U_l, right U_r and two-sided U_t uniformities.

A topological group G is said to be

(1) *M-bounded if for each sequence $(U_n | n \in N)$ of neighborhoods of the neutral element $e \in G$ there is a sequence $(A_n | n \in N)$ of finite subsets of G such that $G = \bigcup\limits_{n \in N} A_n \cdot U_n$ [3];*

*(**2**) H-bounded if for each sequence $(U_n|n \in N)$of neighborhoods of the neutral element $e \in G$ there is a sequence $(A_n|n \in N)$ of finite subsets of G such that each $x \in G$ belongs to all but finitely many $A_n \cdot U_n$ [3];*

*(**3**) R-bounded if for each sequence $(U_n|n \in N)$ of neighborhoods of the neutral element $e \in G$ there is a sequence $(A_n|n \in N)$ of elements of G such that $G = \bigcup_{n \in N} x_n \cdot U_n$ [3].*

In other words topological group G is said to be M-bounded (H-bounded, R-bounded), if G a uniformly Menger (uniformly Hurewicz, uniformly Rothberger) space with respect to the left or right group uniformity.

Let G be a topological group.

Proposition 1. *Any totally bounded topological group G is M-bounded.*

Corollary 1. *Any compact topological group G is M-bounded.*

Proposition 2. *Any M-bounded topological group G is ω-bounded.*

Theorem 1.

Theorem 1. *If a topological group H is a homomorphic image of a M-bounded (H-bounded, R-bounded) topological group G, then H is also M-bounded (H-bounded, R-bounded) topological group.*

A topological group G is called σ-totally bounded, if it can be represented as the union of countably many totally bounded subspaces.

Results

Proposition 3. *Any* σ*-totally bounded topological group is* H*-bounded.*

Corollary 2. *Any* σ*-totally bounded topological group is* M*-bounded.*

Theorem 2.

Theorem 2. *Any countable discrete topological group is a* R*-bounded.*

Theorem 3.

Theorem 3. *For a locally compact topological group* G *the following are equivalent:*

(1) G*-* M*-bounded;*

(2) G*-* ω*-bounded.*

Results

Proposition 4. *The completion of M-bounded (H-bounded, R-bounded) topological group is a M-bounded (H-bounded, R-bounded).*

Let $f : G \to H$ be a homomorphism of a topological group G onto a topological group H. The mapping f is called precompact, if the mapping $f : (G, U) \to (H, U')$ of a uniform space (G, U) to uniform space (H, U') is precompact.

Theorem 4.

Theorem 3.4. *Let $f : G \to H$ be a precompact mapping. If topological group H is H-bounded, then a topological group G is also H-bounded.*

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