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On Countably Uniformly Paracompact Mappings

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Abstract. In this work we introduce and study countably uniformly paracompact mappings.
Keywords: Uniformly continuous mapping, countably uniformly paracompact mappings, countably uniformly paracompact space.
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INTRODUCTION

Throughout the work all uniform spaces are assumed to be Hausdorff and mappings continuous.

For covers λ and μ of a set X, we have:

$$\lambda \wedge \mu = \{L \bigcap K : L \in \lambda, K \in \mu\};$$
$$\lambda(x) = \bigcup St(\lambda, x), St(\lambda, x) = \{L \in \lambda : L \ni x\}, x \in X;$$
$$\lambda(H) = \bigcup St(\lambda, H), St(\lambda, H) = \{L \in \lambda : L \bigcap H \neq \varnothing\}, H \subset X.$$

For covers λ and μ of the set X, the symbol $\lambda \succ \mu$ means that the cover λ is a refinement of the cover μ, i.e. for any $L \in \lambda$ there is $K \in \mu$ such as $L \subset K$ and, the symbol $\lambda * \succ \mu$ means that the cover λ is a strongly star refinement of the cover μ, i.e. for any $L \in \lambda$ there is $K \in \mu$ such as $\lambda(L) \subset K$. The cover λ is finitely additive if $\lambda = \lambda^{\angle}$, $\lambda^{\angle} = \{\bigcup \lambda_0 : \lambda_0 \subset \lambda \text{ is finite }\}$. Let $g : X \to Y$ be a mapping. If λ and μ are the covers of X and Y, respectively, then $g\lambda = \{gL : L \in \lambda\}$ and $g^{-1}\mu = \{g^{-1}K : K \in \mu\}$ is covers of Y and X, respectively. $g^{\#}\lambda = \{g^{\#}L : L \in \lambda\}$, $g^{\#}L = Y \backslash g(X \backslash L)$. For cover some properties see [2], [5], [7].

A cover λ of the uniform space (X, Σ) is called *uniformly locally finite*, if there is a cover $\mu \in \Sigma$ such that every $K \in \mu$ meets λ only for a finite number of elements of λ, [3].

A uniform space (X, Σ) is called *countable uniformly paracompact* if every countable open cover λ has a uniformly locally finite refinement, [4];

Let $g : (X, \Sigma) \to (Y, \Sigma')$ be a uniformly continuous mapping of a uniform space (X, Σ) onto a uniform space (Y, Σ'). The mapping g is called:

1) *uniformly open*, if g maps each open cover $\alpha \in \Sigma$ to an open cover $g\alpha \in \Sigma'$ [1];
2) *precompact*, if for each $\lambda \in \Sigma$ there is a cover $\mu \in \Sigma'$ and finite cover $\gamma \in \Sigma$, such that $g^{-1}\mu \wedge \gamma \succ \lambda$ [1];
3) *uniformly perfect*, if it is both precompact and perfect [1].

For the uniformity Σ by τ_{Σ} we denote the topology generated by the uniformity .

COUNTABLE UNIFORMLY PARACOMPACT SPACE AND MAPPINGS

Definition 1 *Let $g : (X, \Sigma) \to (Y, \Sigma')$ be a uniformly continuous mapping of the space (X, Σ) onto a uniform space (Y, Σ'). The mapping g is said to be countably uniformly paracompact if for each countable open cover λ of (X, Σ) there exist a countable open cover μ of (Y, Σ') and a uniformly locally finite open cover γ of (X, Σ), such that $g^{-1}\mu \wedge \gamma \succ \lambda$.*

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Proposition 1 *Let $g : (X, \Sigma) \to (Y, \Sigma^{'})$ be a uniformly continuous mapping of a uniform space (X, Σ) onto a uniform space $(Y, \Sigma^{'})$. If (X, Σ) is countably uniformly paracompact then the mapping g is countably uniformly paracompact.*

Proof. Let (X, Σ) be a countably uniformly paracompact space and α be an arbitrary open cover. Then exist uniformly locally finite open cover λ of (X, Σ) such as λ is refinement of α. For countable open cover μ of the space $(Y, \Sigma^{'})$ we have that the cover $g^{-1}\mu \wedge \lambda \succ \alpha$. Consequently, the mapping g is countably uniformly paracompact.

Proposition 2 *If a uniformly continuous mapping $g : (X, \Sigma) \to (Y, \Sigma^{'})$ of a uniform space (X, Σ) onto a uniform space $(Y, \Sigma^{'})$, $Y = \{y\}$ is countably uniformly paracompact, then the space (X, Σ) is countably uniformly paracompact.*

Proof. Let g be a countably uniformly paracompact mappings and α be an arbitrary open cover of the space (X, Σ). Then exists such countable open cover μ of the space $(Y, \Sigma^{'})$ and uniformly locally finite open cover λ of the space (X, Σ), that the cover $g^{-1}\mu \wedge \lambda \succ \alpha$. Since $Y = \{y\}$, then $f^{-1}\mu \wedge \lambda = \lambda$. Thus, (X, Σ) is countably uniformly paracompact.

Lemma 1 *If λ and μ are uniformly locally finite covers of the space (X, Σ), then $\lambda \wedge \mu$ is a uniformly locally finite cover of (X, Σ).*

Proof. Let λ and μ be uniformly locally finite covers of the space (X, Σ). We show the cover $\lambda \wedge \mu$ is also a uniformly locally finite cover of (X, Σ). Since the covers λ and μ are uniformly locally finite, there are covers $\beta \in \Sigma$ and $\eta \in \Sigma$, such as $M \subset \bigcup_{i=1}^{n} L_i$, $N \subset \bigcup_{j=1}^{m} K_j$ for any $M \in \beta$, $N \in \eta$. Note that $M \bigcap N \subset \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} (L_i \bigcap K_j)$, $M \bigcap N \in \beta \wedge \eta$. Obviously, $\beta \wedge \eta \in \Sigma$. Thus, $\lambda \wedge \mu$ is uniformly locally finite.

Lemma 2 *Let $g : (X, \Sigma) \to (Y, \Sigma^{'})$ be a uniformly continuous mapping of the space (X, Σ) onto the space $(Y, \Sigma^{'})$. If μ is uniformly locally finite open cover of the space $(Y, \Sigma^{'})$, then $g^{-1}\mu$ is uniformly locally finite open cover of the space (X, Σ).*

Proof. Let g be a uniformly continuous mapping and μ be a uniformly locally finite open cover of the space $(Y, \Sigma^{'})$. Then there is a cover $\lambda \in \Sigma^{'}$ such as for any $L \in \lambda$ there are elements $K_i \in \mu$, $i = 1, 2, ..., n$, such that $L \subset \bigcup_{i=1}^{n} K_i$. Since g is a uniformly mapping, then $g^{-1}\mu$ is an open cover of (X, Σ) and $g^{-1}\lambda \in \Sigma$. Consequently, $g^{-1}L \subset \bigcup_{i=1}^{n} g^{-1}K_i$, $g^{-1}K_i \in g^{-1}\mu$. Thus, the cover $g^{-1}\mu$ is a uniformly locally finite open cover of the space (X, Σ).

Proposition 3 *Let $g : (X, \Sigma) \to (Y, \Sigma^{'})$ be a uniformly continuous mapping of the space (X, Σ) onto the space $(Y, \Sigma^{'})$ and $j : (Y, \Sigma^{'}) \to (Z, \Sigma^{''})$ be a uniformly continuous mapping of the space $(Y, \Sigma^{'})$ onto the space $(Z, \Sigma^{''})$. If g and j are countably uniformly paracompact, then their composition $h = j \circ g$ is countably uniformly paracompact.*

The proof follows from Lemmas 1 and 2.

Proposition 4 *If $g : (X, \Sigma) \to (Y, \Sigma^{'})$ is a countably uniformly paracompact mapping $M \subset X$ is a closed subset, then its restriction $g|_M : (M, \Sigma_M) \to (Y, \Sigma^{'})$ is a countably paracompact mapping.*

Proof. Let λ_M be an arbitrary finitely additive countably open cover of the space (M, Σ_M). Then there are finitely additive countably open family α of the space (X, Σ), such that $\alpha \wedge \{M\} = \lambda_M$. It is clear that the family $\lambda = \{\alpha, X \backslash M\}$ is a finitely additive countably open cover of (X, Σ). Then there is countably open cover μ of $(Y, \Sigma^{'})$ and a uniformly locally finite cover $\gamma \in \Sigma$, such that $g^{-1}\mu \wedge \gamma \succ \lambda$. It is easy to see $(g|_M)^{-1}\mu \wedge \gamma_M \succ \lambda_M$. Therefore, the mapping $g|_M$ is countably uniformly paracompact.

Theorem 1 *If $g : (X, \Sigma) \to (Y, \Sigma^{'})$ is a countably uniformly paracompact mapping, and $(Y, \Sigma^{'})$ is countably uniformly paracompact, then the uniform space (X, Σ) is countably uniformly paracompact.*

Proof. Let α be an arbitrary open cover of the space (X, Σ). Then there is an open cover μ of the space $(Y, \Sigma^{'})$ and a uniformly locally finite open cover λ of the space (X, Σ), such that $g^{-1}\mu \wedge \lambda \succ \alpha$. By countably uniform paracompactness of the space $(Y, \Sigma^{'})$, [5]. There is uniformly locally finite open cover μ_0, such as the cover μ_0 is a refinement of μ. Obviously, $g^{-1}\mu_0 \wedge \lambda \succ g^{-1}\mu \wedge \lambda$. By virtue of Lemma 2 the open cover $g^{-1}\mu_0$ is uniformly star finite. Put $g^{-1}\mu_0 \wedge \gamma = \delta$. By virtue of Lemma 1 the open cover δ is uniformly locally finite. Hence, the space (X, Σ) is countably uniformly paracompact, [12].

The question arises: Which mappings preserve countable uniform paracompactness in both directions?

Theorem 2 *Let $g : (X, \Sigma) \to (Y, \Sigma')$ be uniformly perfect mapping between the space (X, Σ) and (Y, Σ'). Then uniformly countable paracompactness is preserved both in the image and the preimage direction.*

Proof. Let the space (X, Σ) be a countably uniformly paracompact and μ be an arbitrary finitely additive countable open cover of the space (Y, Σ'), [7]. Then $g^{-1}\mu$ is a finitely additive countable open cover of the space (X, Σ). By Condition 2 in [4] $g^{-1}\mu \in \Sigma$. In view of the precompactness of the mapping g there are a cover $\alpha \in \Sigma'$ and a finite cover $\gamma \in \Sigma$, such that $g^{-1}\alpha \wedge \gamma \succ f^{-1}\mu$. Note that $(g^{-1}\alpha \wedge \gamma)^{\angle} = (g^{-1}\alpha)^{\angle}$ and $(g^{-1}\beta)^{\angle} = g^{-1}\mu^{\angle}$. Therefore $(g^{-1}\alpha)^{\angle} \succ g^{-1}\mu$ hence $\alpha \succ \mu$. Then $\beta \in \Sigma'$. By Condition 2 in [4] the space (Y, Σ') is countably uniformly paracompact, [5].

Conversely, let the space (Y, Σ') be a countably uniformly paracompact and λ be an arbitrary finitely-additive countable open cover of (X, Σ). Then $\{g^{-1}y : y \in Y\} \succ \lambda$, [5]. Since g is a closed mapping, it follows that $\alpha = \{g^{\#}L : L \in \lambda\}$ is finitely-additive countable open cover of (Y, Σ'), $g^{\#}L = Y \backslash f(X \backslash L)$. By countable paracompactness of the space (Y, Σ') there exist a locally finite cover $\mu \in \Sigma'$, such that $\mu \succ \alpha$. Then $g^{-1}\mu \succ \alpha$ and according to Lemma 2 the cover $g^{-1}\mu$ is locally finite. Consequently, (X, Σ) is countably uniformly paracompact.

Theorem 3 *Let $g : (X, \Sigma) \to (Y, \Sigma')$ be a uniformly open mapping of the space (X, Σ) onto the space (Y, Σ'). If (X, Σ) is countably uniformly paracompact space, then (Y, Σ') is countably uniformly paracompact.*

Proof. Let g be a uniformly open mapping of (X, Σ) onto (Y, Σ') and λ be an arbitrary open countable cover of (X, Σ), [12]. Then $g^{-1}\lambda = \alpha$ is an open countable cover of the space (X, Σ). Since (X, Σ) is countably uniformly paracompact, then by Condition 2 in [4] we have $\mu^{\angle} \in \Sigma$. Note that $g(\gamma^{\angle}) = \lambda^{\angle}$. Then from the uniform openness of the mapping g we have $\lambda^{\angle} \in \Sigma'$. Consequently, the space (Y, Σ') (by Condition 2 in [4]) is countably uniformly paracompact.

Recall that a uniformly continuous mapping $g : (X, \Sigma) \to (Y, \Sigma')$ of uniform space (X, Σ) onto a uniform space (Y, Σ') is called uniformly paraclosed, if for any open uniform cover $\lambda \in \Sigma$ of the space (X, Σ), the cover $\left\{f^{-1}y : y \in Y\right\}$ is a refinement of $\lambda^{\angle}$. Then $g^{\#}\lambda^{\angle}$ is an open uniform cover of (Y, Σ'). A uniformly continuous mapping $g : (X, \Sigma) \to (Y, \Sigma')$ of the space (X, Σ) onto the space (Y, Σ') is called uniformly paraperfect if it is both paraclosed and compact, [1].

Proposition 5 *If a uniformly continuous mapping $g : (X, \Sigma) \to (Y, \Sigma')$ of the space (X, Σ) onto the space (Y, Σ') is uniformly paraperfect, then the continuous mapping $g : (X, \tau_{\Sigma}) \to (Y, \tau_{\Sigma'})$ of the topological space (X, τ_{Σ}) onto the topological space $(Y, \tau_{\Sigma'})$ is a closed mapping.*

Proof. Let O be an arbitrary open cover of the space (X, τ_{Σ}) such that $O \supset g^{-1}y$, [6]. Then there exist $\lambda \in \Sigma$ such that $\lambda(g^{-1}y) \subset O$. It is clear that the cover $\left\{g^{-1}y : y \in Y\right\}$ is a refinement of $\lambda^{\angle}$. Since the mapping g is paraperfect, then $g^{\#}\lambda^{\angle} \in \Sigma'$. Assume $\beta = f^{\#}\lambda^{\angle}$. Then there exists $B \in \beta$ such that $B \ni y$ and $g^{-1}B \subset O$. Hence the continuous mapping $g : (X, \tau_{\Sigma}) \to (Y, \tau_{\Sigma'})$ is a closed.

Proposition 6 *Let $g : (X, \Sigma) \to (Y, \Sigma')$ be a uniformly paraperfect mapping. If (X, Σ) is countably uniformly paracompact then (Y, Σ') is also countably uniformly paracompact.*

Proof. Let μ be an arbitrary finite additive countable open cover of (Y, Σ'). Then $g^{-1}\mu = \lambda$ is finitely additive countable open cover of (X, Σ), [7]. It is easy to see that the cover $\left\{g^{-1}y : y \in Y\right\}$ consisting of compact subsets $g^{-1}y$ of the space (X, Σ) is a refinement of λ. Then, due to the uniform paraperfecness of the mapping g, the cover $g^{\#}\lambda^{\angle} = g^{\#}\lambda$ is an open cover of the space (Y, Σ'). It is clear that $g^{\#}\lambda \succ \mu$. Therefore, μ is a uniform cover of (Y, Σ'). So (Y, Σ') is a countably uniformly paracompact space.

Theorem 4 *Let $g : (X, \Sigma) \to (Y, \Sigma')$ be a uniformly paraperfect mapping. Then (X, Σ) is a countably uniformly paracompact iif the (Y, Σ') is countably uniformly paracompact.*

Proof. Let (Y, Σ') be a countably uniformly paracompact space and λ be an arbitrary finitely additive open cover of (X, Σ). Note that the cover $\left\{g^{-1}y : y \in Y\right\}$ is a refinement of the cover λ. According to Proposition 5 the cover $g^{\#}\lambda$ is open, put $\mu = g^{\#}\lambda$. Since the uniform space (Y, Σ') is paracompact, then cover $\mu^{\angle} = g^{\#}\lambda^{\angle} = g^{\#}\lambda$ is a uniform cover of the space (Y, Σ'). It is easy to see that $g^{-1}\mu^{\#} \succ \lambda$, [10]. Hence (X, Σ) is countably uniformly paracompact. The validity of the converse assertion follows from Proposition 6.

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