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On the *R*-Compactification of Uniform Spaces

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Abstract. As it is well known, there are various constructions of the *R*-compactification (Hewitt real compactification) of a uniform space [13], [15]. In this work we propose a new construction of the *R*-compactification (Hewitt real compactification) of a uniform space.
Keywords: $\aleph_0$-boundedness, *R*-compactification, *R*-extension, *R*-completeness.
MSC: 54E15, 54D20.

INTRODUCTION

In this paper, it is presupposed that all uniform spaces are Hausdorff. Consequently, the mappings within these spaces are uniformly continuous.

For systems λ and μ of a set X, we have [11]:

$$\lambda \wedge \mu = \{L \bigcap M : L \in \lambda, M \in \beta\}.\ \lambda(x) = \bigcup St(\lambda, x);$$
$$St(\lambda, x) = \{L \in \lambda : L \ni x\},\ x \in X;$$
$$\lambda(H) = \bigcup St(\lambda, H),\ St(\lambda, H) = \{L \in \lambda : L \bigcap H \neq \varnothing\},\ H \subset X.$$

For coverings λ and μ of the set X, the symbol $\lambda \succ \mu$ means that the covering λ is a refinement of the covering μ, i.e. for any $L \in \lambda$ there are $M \in \mu$ such as $L \subset M$, and the symbol $\lambda * \succ \mu$ denote that the covering λ is a strongly star refinement of μ, i.e. for any $L \in \lambda$ there are $M \in \mu$ such as $\lambda(L) \subset M$. Let $g : X \to Y$ be a mapping. If λ and μ are the coverings of X and Y, respectively, then $g\lambda = \{gL : L \in \lambda\}$ and $g^{-1}\mu = \{g^{-1}M : M \in \mu\}$ are coverings of Y and X, respectively, [6].

A Tychonoff space X is called *R*-complete, if there is no Tychonoff space X^* that satisfies the following two conditions:

$(RC1)$ An embedding exists that is homeomorphic to $r : X \to X^*$ such as $r(X) \neq [r(X)]_{X^*} = X^*$.

$(RC2)$ For any continuous real mapping $g : X \to R$ there is a mapping $g^* : X^* \to R$ such as $g^* r = g$, [13].

Asume that X is a nonempty set. If the following conditions are met, the system Σ of coverings of set X is referred to as a uniformity on X:

$(P1)$ If $\lambda \in \Sigma$ and λ is a refinement of the covering μ of the set X, then $\mu \in \Sigma$;

$(P2)$ For any $\lambda_1 \in \Sigma$ and $\lambda_2 \in \Sigma$ there are $\lambda \in \Sigma$ such as $\lambda \succ \lambda_1$ and $\lambda \succ \lambda_2$;

$(P3)$ For any $\lambda \in \Sigma$ there are $\mu \in \Sigma$ such as $\mu * \succ \lambda$;

$(P4)$ For any pair of different points $x, y \in X$ there are $\lambda \in \Sigma$ such as no element λ contains both x and y.

For the uniformity Σ by τ_Σ we denote the topology generated by the uniformity. When a uniformity Σ on a set X produces the same topology as X, we can say that Σ is a uniformity on the topological space X. The filter F is called a Cauchy filter in (X, Σ) if $\lambda \bigcap F \neq \varnothing$ for any $\lambda \in \Sigma$, [9].

A uniform space (X, Σ) is called:

(i) precompact if the uniformity Σ has a base that comprises finite coverings, [1], [9];

(ii) the uniformity Σ is considered $\aleph_0$-bounded if it has a base comprising countable coverings, [3].

(iii) complete if every Cauchy filter in it converges, [1], [2], [4].

A uniform space (X^*, Σ^*) is called the completion of a uniform space (X, Σ), if

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1. $X \subset X^*$;
2. (X,Σ) is everywhere dense in (X^*,Σ^*);
3. (X^*,Σ^*) - complete, [5].

The completion (X^*,Σ_c^*) of a uniform space (X,Σ_c) is called the Samuel compactification of (X,Σ), where Σ_c is the precompact reflection of the uniformity Σ (see [1]). The notation $(sX, s\Sigma)$ represents the Samuel compactification of a uniform space (X,Σ), [5], [10].

A mapping $g : (X,\Sigma) \to (Y,\Sigma^{'})$ of a space (X,Σ) into a space $(Y,\Sigma^{'})$ is called uniformly continuous if for any $\mu \in \Sigma^{'}$ there are $\lambda \in \Sigma$ such as $g\lambda \succ \mu$, [1], [2].

Suppose $g : (X,\Sigma) \to (Y,\Sigma^{'})$ is a uniformly continuous mapping from a space (X,Σ) to a space $(Y,\Sigma^{'})$. A bijective mapping $g : (X,\Sigma) \to (Y,\Sigma^{'})$ of a space (X,Σ) to a space $(Y,\Sigma^{'})$ is called a uniform homeomorphism or a uniform isomorphism if both $g : (X,\Sigma) \to (Y,\Sigma^{'})$ and $g^{-1} : (Y,\Sigma^{'}) \to (X,\Sigma)$ are uniformly continuous. For each space (X,Σ) and any subspace (X_0,Σ_{X_0}), the formula $j_{X_0}(x) = x$ defines a uniformly mapping $j_{X_0} : (X_0,\Sigma_{X_0}) \to (X,\Sigma)$. The mapping j_{X_0} is called a uniform embedding of the subspace (X_0,Σ_{X_0}) into the space (X,Σ), [1], [2], [4].

R-COMPACTIFICATION OF UNIFORM SPACE

Let (X,Σ) be a uniform space.

Lemma 1 *A uniform space (X,Σ) is $\aleph_0$-bounded iff every uniform covering contains a countable subcovering.*

Proof. Suppose $\lambda \in \Sigma$ is an arbitrary covering. Since (X,Σ) is $\aleph_0$-bounded, there is a countable uniform covering $\mu \in \Sigma$ such that $\mu \succ \lambda$. For any $M_n \in \mu$ we choose one $L_{M_n} \in \lambda$ such that $M_n \subset L_{M_n}$. Put $\lambda_0 = \{L_{M_n}, n \in N\}$. Then $\lambda_0 \subset \lambda$ is a countable subcovering, [5].

Conversely, let $\lambda \in \Sigma$ be an any covering and $\mu, \eta \in \Sigma$ are uniform coverings such that $\eta * \succ \mu$ and $\mu * \succ \lambda$. Let η_0 be a countable subcovering of η, [5]. For each $E \in \eta_0$ choose one element $x_E \in E$ and put $\mathrm{G} = \{x_E : E \in \eta_0\}$. It is clear that G is a countable subset of the space (X,Σ). Let $E \in \eta$ be an any element and $y \in E$ be an arbitrary selected point. Then there is $x_E \in \mathrm{G}$ such that $y \in \eta(x_E)$. There is $M \in \mu$ such that $x_E \in \eta(y) \subset \eta(E) \subset M$. It follows from this, that $\eta(E) \subset \mu(x_E)$. Now, for x_E choose one $L_{x_E} \in \lambda$ such that $\mu(x_E) \subset L_{x_E}$. Let $\lambda_0 = \{L_{x_E} : x_E \in \mathrm{G}\}$. Then $\eta * \succ \lambda_0$. Therefore, $\lambda_0 \in U$. So (X,U) is $\aleph_0$-bounded.

Theorem 1 *Let (X,Σ) be an arbitrary uniform space. Then there exists a $\aleph_0$-bounded uniformity Σ_l on X, satisfying the following conditions:*

1) $\Sigma_l \subset \Sigma$;

2) the topologies generated by Σ_l and Σ coincide;

3) Σ_l is the largest $\aleph_0$-bounded uniformity contained in Σ.

Proof. Let $\Sigma_l = \{\lambda \in \Sigma :$ there are a countable covering $\mu \in \Sigma$ which refines $\lambda\}$. We verify that all the conditions of uniformity are satisfied. Let us check the condition $(P1)$. Let $\lambda \in \Sigma_l$ and $\lambda \succ \mu$. Then there are a countable covering $\gamma \in \Sigma$ which is a refinement of λ. Then, $\mu \in \Sigma_l$. Condition $(P1)$ is fulfilled. Now we check the condition $(P2)$. Let $\lambda_1, \lambda_2 \in \Sigma_l$. Then there exist countable covering $\mu_1, \mu_2 \in \Sigma$, such that $\mu_1 \succ \lambda_1$ and $\mu_2 \succ \lambda_2$. Put $\mu = \mu_1 \wedge \mu_2$. Note that μ is a countable covering and $\mu \succ \lambda_1 \wedge \mu_2$. By the definition of Σ_l we have that $\lambda_1 \wedge \lambda_2 \in \Sigma_l$. Put $\lambda = \lambda_1 \wedge \lambda_2$. Then $\lambda \succ \lambda_1$ and $\lambda \succ \lambda_2$. Let $\lambda \in \Sigma_l$. Then there are a countable covering $\mu \in \Sigma$ which is a refinement of λ. Let $\gamma \in \Sigma$ be a uniform covering such as $\gamma * \succ \mu$. Then from Exercise 8.1.I. in [5], the condition $(P3)$ is satisfied. Let $x, y \in X$ are distinct points. Then there are a covering $\lambda \in \Sigma$ such as $y \notin \lambda(x)$. If $\mu \in \Sigma$ is a countable covering such as $\mu \succ \lambda$, then $y \notin \mu(x)$. Hence, condition $(P4)$ is also satisfied. It is easy to see that $\tau_\Sigma = \tau_{\Sigma_l}$. By the construction, Σ_l is the greatest uniformity contained in Σ.

A uniformity Σ_l in the previous theorem is called the $\aleph_0$-bounded reflection of the Σ, and the completion (X^*,Σ_l^*) of the uniform space (X,Σ_l) is called the *R-compactification of the uniform space* (X,Σ). The *R*-compactification of a uniform space (X,Σ) is denoted by $(\nu X, \nu\Sigma)$. If Σ_X is the universal uniformity space X, then $(\nu X, \nu\Sigma)$ is the *R*-extension of the space X.

Σ_l be a $\aleph_0$-bounded reflection of the uniformity Σ, and $j_X : (X,\Sigma_l) \to (X,\Sigma_l)$ be a canonical uniform embedding, $j_X(x) = x$ for any $x \in X$. Since $\Sigma_l \subset \Sigma$ and $(X^*,\Sigma_l^*) = (\nu X, \nu\Sigma)$, then the canonical injection $j_X : (X,\Sigma) \to (\nu X, \nu\Sigma)$ is uniformly continuous.

Theorem 2 *Let $g : (X,\Sigma) \to (Y,\Sigma^{'})$ be a uniformly continuous mapping of (X,Σ) onto $(Y,\Sigma^{'})$, [7]. Then there is a unique uniformly continuous mapping "onto" $v(g) : (vX, v\Sigma) \to (vY, v\Sigma^{'})$ such that $v(g) \circ j_X = j_Y \circ g$.*

Theorem 3 *For any space (X,Σ) there exist exactly one (up to a uniform isomorphism) R-complete space $(\nu X, \nu\Sigma)$ with the following properties:*

1. There exist a uniformly isomorphism $j : (X,\Sigma) \to (vX, v\Sigma)$ for which $[j(X)]_{\nu X} = \nu X$.

2. For any uniformly mapping $g : (X,\Sigma) \to (R,\Sigma_R)$, there exists a uniformly continuous mapping $\hat{g} : (\nu X, \nu\Sigma) \to (R,\Sigma_R)$ such as $\hat{g} \circ j = g$.

Proof. Let $g : (X,\Sigma) \to (R,\Sigma_R)$ be a uniformly mapping of a space (X,Σ) to a space (R,Σ_R). Consider the extension $\hat{g} : (sX, s\Sigma_X) \to (\lambda R, \Sigma^{'})$ of the mapping g, where $(sX, s\Sigma_X)$ is the Samuel compactification (see [14]) with respect to universal uniformity, $(\lambda R, \lambda\Sigma_R)$ is the one-point compactification (see [14]) of the space (R,Σ_R). Let $N_g = \hat{g}^{-1}(R)$. Clearly that $X \subset N_g$. Denote by F the set of all continuous real functions on X. Put $\nu X = \bigcap_{g\in F} N_g$. The mapping $j : (X,\Sigma) \to (\nu X, \nu\Sigma)$, $j(x) = s(x)$ as $x \in X$, is a uniform isomorphism. Therefore, condition 1. is satisfied. It follows from the construction that $(\nu X, \nu\Sigma)$ satisfies condition 2. The R-completeness of the space $(\nu X, \nu\Sigma)$ follows from the fact that a product is R-complete iff each factor of the product is R-complete.

Lemma 2 *Let (X,Σ) be a space and (X_0, Σ_{X_0}) its dense subspace, and $g_0 : (X_0, \Sigma_{X_0}) \to (R,\Sigma_R)$ be a uniformly mapping of (X_0, Σ_{X_0}) into (R,Σ_R). Then there exist a unique uniformly mapping $g : (X,\Sigma) \to (R,\Sigma_R)$ such as $g|_{X_0} = g_0$.*

Proof. Suppose $x \in X$ is an any point, and B_x be a neighborhood filter for x in (X,Σ), [3]. Put $F_x = B_x \bigcap \{X_0\}$. Then F_x is Cauchy filter in (X_0, Σ_{X_0}). It is clear that $g(F_x)$ is the base of some Cauchy filter F_R in (R,Σ_R). Let $y \in R$ be the limit point. Put $g(x) = y$, [7]. Thus, the mapping $g : X \to R$ of the set X to R is defined. We show that g is uniformly continuous. Let $\lambda \in \Sigma_R$ be a covering. Choose a covering $\mu \in \Sigma_R$ such that $\mu* \succ \lambda$. Then there exists $\lambda_0 \in \Sigma_{X_0}$ such that $g_0\lambda_0 \succ \mu$. We show that $g[\lambda_0] \succ [\mu]$, where $[\lambda_0] = \{|L_0|_X : L_0 \in \lambda_0\}$. Let $[L_0]_X \in [\lambda_0]$. If $x \in [L_0]_X$, then by the definition of the mapping g, $g(x) \in [g(L_0)]_R$. Since $g_0\lambda_0* \succ \lambda$, then $\mu(g(L_0)) \subset L$ for some $L \in \lambda$. Therefore, $[g(L_0)]_R \subset L$. So, $g[\lambda_0] \succ \lambda$. According to the definition of the mapping g we have that $g|_{X_0} = g_0$. The uniqueness of g follows from the fact that any two continuous mappings defined on a Hausdorff space coinciding on a dense subspace coincide on the entire space.

Lemma 3 *Let (X,Σ) be a space and (X_0, Σ_{X_0}) its subspace. If every uniformly mapping $g : (X,\Sigma) \to (R,\Sigma_R)$ can be extended uniformly continuous to (X,Σ), then every uniformly continuous mapping $g : (X_0, \Sigma_{X_0}) \to \prod_{i\in I}(R_i, \Sigma_{R_i})$ to the product of copies of the real line also can be extended uniformly continuous to (X,Σ). If (X_0, Σ_{X_0}) is dense in (X,Σ), then every uniformly mapping of $g : (X_0, \Sigma_{X_0}) \to (Y,\Sigma^{'})$, $Y = [Y] \subset \prod_{i\in I}(R_i, \Sigma_{R_i})$ into a closed subspace $(Y,\Sigma^{'})$ of such a product extends to a uniformly mapping of (X,Σ) into $(Y,\Sigma^{'})$, [7], [12].*

Proof. Let $g : (X_0, \Sigma_{X_0}) \to \prod_{i\in I}(R_i, \Sigma_{R_i})$ be a uniformly continuous mapping, where $R_i = R$ for any $i \in I$. For each $i \in I$ the extension $\hat{g}_i : (X,\Sigma) \to (R_i, \Sigma_{R_i})$ of the composition $p_i \circ g : (X_0, \Sigma_{X_0}) \to (R_i, \Sigma_{R_i})$ is defined. Obviously, [6] that the diagonal mapping $\Delta_{i\in I} g_i : (X,\Sigma) \to \prod_{i\in I}(R_i, \Sigma_{R_i})$ is an extension of the mapping $g : (X_0, \Sigma_{X_0}) \to \prod_{i\in I}(R_i, \Sigma_{R_I})$. If (X_0, Σ_{X_0}) is dense in (X,Σ), then for the extension $\Delta_{i\in I} g_i : (X,\Sigma) \to \prod_{i\in I}(R_i, \Sigma_{R_I})$ of the mapping $j_Y \circ g : (X_0, \Sigma_{X_0}) \to \prod_{i\in I}(R_i, \Sigma_{R_i})$ we have $\Delta_{i\in I} g_i(X) = \Delta_{i\in I}[X_0] \subset \left[\Delta_{i\in I} X_0\right] \subset [Y] = Y$. Therefore, the mapping $\Delta_{i\in I} g_i|_Y : (X,\Sigma) \to (Y,\Sigma^{'})$ is an extension of the mapping g, [5].

Theorem 4 *For every uniformly mapping $g : (X,\Sigma) \to (Y,\Sigma^{'})$ of a space (X,Σ) to any R-complete space (X,Σ), there are an uniformly mapping $\hat{g} : (\nu X, \nu\Sigma) \to (Y,\Sigma^{'})$ such as $\hat{g} \circ j = g$, where $j : (X,\Sigma) \to (\nu X, \nu\Sigma)$ is a uniformly isomorphic embedding.*

The proof follows from Theorem 3.11.3 in [13], Theorem 3, and Lemmas 2 and 3.

The uniqueness of the space $(\nu X, \nu\Sigma)$ follows from Theorems 3 and 4.

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