About Weakly Uniformly Paracompact Spaces

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Introduction

Throughout this work all uniform spaces are assumed to be Hausdorff, topological space Tychonoff and mappings are uniformly continuous.
For coverings α and β of the set X, the symbol $\alpha \succ \beta$ means that the covering α is a refinement of the covering β, i.e. for any $A \in \alpha$ there exist $B \in \beta$ such that $A \subset B$. The covering α is called finite additive, if $\alpha^{\angle} = \alpha$, $\alpha^{\angle} = \{\bigcup \alpha_0 : \alpha_0 \subset \alpha \text{ is finite}\}$. A uniformly continuous mapping $f : (X, U) \to (Y, V)$ of a uniform space (X, U) to a uniform space (Y, V) is called precompact, if for each $\alpha \in U$ there exist a uniform covering $\beta \in V$ and a finite uniform covering $\gamma \in U$, such that $f^{-1}\beta \wedge \gamma \succ \alpha$ [4] ;

a uniformly continuous mapping $f : (X, U) \to (Y, V)$ of a uniform space (X, U) into a uniform space (Y, V) is called uniformly perfect if it is both precompact and perfect [4]; Let ω be an open cover of a topological space X to the topological space Y. A mapping f is called an ω-mapping if every point $y \in Y$ has a neighborhood O_y whose inverse image $f^{-1}O_y$ is contained in at least one element of the cover ω [1]; a cover α of a topological space X is called point-finite if every point of X lies in only finitely many members of α [3]; a uniform space (X, U) is called uniformly A-paracompact if every of its finitely additive open cover has a locally finite uniform refinement [2]. For the uniformity U by τ_U we denote the topology generated by the uniformity and symbol U_X means the universal uniformity.

Weakly Uniformly Paracompactness

Let (X, U) be a uniform space.

Definition 1

A uniform space (X, U) is called weakly uniformly paracompact if every finitely additive open cover has a point-finite uniform refinement.

Proposition 1

If (X, U) is a weakly uniformly paracompact space, then the topological space (X, τ_U) is weakly paracompact. Conversely, if (X, τ) is weakly paracompact, then the uniform space (X, U_X), where U_X is the universal uniformity, is weakly uniformly paracompact.

Proof. Let α be an arbitrary open covering of the space (X, τ_U). Then, for a finitely additive open covering $\alpha^{\angle}$ of the uniform space (X, U) there exists a point-finite uniform covering $\beta \in U$ which is a refinement of it. It is known that the interior $\langle\beta\rangle = \{\langle B\rangle : B \in \beta\}$ of a uniform covering of β is a uniform covering, where $\langle B\rangle$ is the interior of the set B. Let $\gamma = \langle\beta\rangle$. It is clear that γ is a point-finite open uniform covering of the (X, U). For each $\Gamma \in \gamma$ choose $A_\Gamma \in \alpha_{\aleph_0}$ such that $\Gamma \subset A_\Gamma$, where $A_\Gamma = \bigcup\limits_{i=1}^{n} A_i$, $A_i \in \alpha$, $i = 1, 2, ..., n$. Let $\alpha_0 = \bigcup\{\alpha_\Gamma : \Gamma \in \gamma\}$, $\alpha_\Gamma = \{\Gamma \bigcap A_i : i = 1, 2, ..., n\}$. Then α_0 is a point-finite open covering of the space (X, τ_U), it is a refinement of the open covering α. So, the space (X, τ_U) is weakly paracompact.

Conversely, let the Tychonoff space (X, τ) be weakly paracompact. Then the set of all open coverings forms the base of the universal uniformity U_X of the space (X, τ). It is easy to see that the uniform space (X, U_X) is weakly uniformly paracompact.
The Japanese mathematician G. Tamano gave a remarkable characteristic of paracompact spaces in terms of compact extensions.
The following theorem gives a characteristic of weakly uniformly paracompactness in the spirit of Tamano.

Theorem 1

Let (X, U) be a uniform space and bX be a certain its compact Hausdorff extension. The uniform space (X, U) is weakly uniformly paracompact, if and only if for each compactum $K \subset bX \backslash X$ there exists a point-finite uniform covering $\alpha \in U$ such that $[A]_{bX} \bigcap K = \varnothing$ for all $A \in \alpha$.

Proof. *Necessity*. Let (X, U) be weakly uniformly paracompact and $K \subset bX \backslash X$ an arbitrary compactum. Then for each point $x \in X$ there is an open neighborhood O_x in bX such that $[O_x]_{bX} \bigcap K = \varnothing$. It is clear that $\gamma = \{O_x \bigcap X : x \in X\}$ is an open covering of the uniform space (X, U). We form an open covering $\gamma^{\angle}$ of the (X, U), taking as elements of γ. Then $\gamma^{\angle}$ is a finite additive open covering of the space (X, U). According to the condition of the theorem, it is possible to refine a covering $\gamma^{\angle}$ by a point-finite uniform covering $\beta \in U$.

Then $[B]_{bX} \subset [\bigcup_{i=1}^{n}(O_{x_i} \bigcap X)]_{bX} \subset \bigcup_{i=1}^{n}[O_{x_i}]_{bX}$. As $[O_{x_i}]_{bX} \bigcap K = \varnothing$ for any $i = 1, 2, ..., n$, then $[B]_{bX} \bigcap K = \varnothing$ for any $B \in \beta$.

Sufficiency. Let α be an arbitrary finite additive open covering of a space (X, U). Then there is an open family β in bX such that $\beta \wedge \{X\} = \alpha$. Let $K = bX \setminus \bigcup \beta$. It follows that K is compactum. Then, by the condition of the theorem, there exists a point-finite uniform covering $\gamma \in U$ such that $[\Gamma]_{bX} \bigcap K = \varnothing$ for any $\Gamma \in \gamma$. Since $[\Gamma]_{bX}$ is compactum in bX there are $B_1, B_2, ..., B_n \in \beta$ such that $[\Gamma]_{bX} \subset \bigcup_{i=1}^{n} B_i$. Then $\Gamma \subset \bigcup_{i=1}^{n} A_i$, where $\bigcup_{i=1}^{n} A_i \in \alpha$. Consequently, (X, U) is a weakly uniformly paracompact space.

Definition 2

A uniform space (X, U) is called uniformly B-locally compact, if there exists a point-finite uniform covering consisting of compact subsets.

The next theorem gives a connection between the weakly uniformly paracompactness and the uniformly B-locally compactness.

Theorem 2

Any uniformly B-locally compact space is weakly uniformly paracompact.

Proof. Let α be an arbitrary finitely additive open covering of the space (X, U). Then there exists a point-finite uniform covering β consisting of compact subsets.

It is easy to see that the covering β is a refinement of the finitely additive open covering α. Consequently, the space (X, U) is weakly uniformly paracompact.
The next two propositions show that weakly uniformly paracompactness is preserved when passing to a closed subspaces and any disjoint sum of uniform spaces.

Proposition 2

Any closed subspace M of a weakly uniformly paracompact space (X, U) is weakly uniformly paracompact.

Proof. Let γ be a finitely additive open covering of M. Let $\hat{\gamma}$ denote the open covering of the space (X, U), there exists of all elements of the covering γ and the set $X \setminus M$. It is clear that $\hat{\gamma}$ is a finitely additive covering. According to the condition there exists a point-finite uniform covering $\beta \in U$ is a refinement $\hat{\gamma}$. Let denote β_M is the trace of β on M.

It is easy to see that β_M is a uniform covering of the subspace M is a refinement γ. Let β_M is a point-finite covering. Indeed, let $x \in M$ be an arbitrary point. Since β is a point-finite uniform covering, then $x \in M \subset X$ belongs to only a finite number of elements of the covering β. Then $x \in M$ belongs to only a finite number of elements of the covering β_M. Thus, in any finitely additive open cover γ of the subspace M, it was possible is a refinement a point-finite uniform covering of β_M. Then, the subspace M is weakly uniformly paracompact.

Proposition 3

The sum of any family of weakly uniformly paracompact spaces is weakly uniformly paracompact.

Proof. Let $\{(X_a, U_a) : a \in M\}$ be an arbitrary family of weakly uniformly paracompact spaces (X_a, U_a) and $(\coprod_{a\in M} X_a, \coprod_{a\in M} U_a)$ is the sum of uniform spaces. Consider an arbitrary finitely additive open covering α of the space $(\coprod_{a\in M} X_a, \coprod_{a\in M} U_a)$. It is easy to see that the family $\beta = \{X_a \bigcap A : a \in M, A \in \alpha\}$ is again a finitely additive open covering of the space $(\coprod_{a\in M} X_a, \coprod_{a\in M} U_a)$ is a refinement α. For each $a_0 \in M$, put $\beta_{a_0} = \{X_{a_0} \bigcap A : a_0 \in M, A \in \alpha\}$.

It is clear that it is a finitely additive open covering of the space (X_{a_0}, U_{a_0}), and therefore, there exists a point-finite uniform covering $\gamma_{a_0} \in U_{a_0}$ is a refinement β_{a_0}. Next, consider the family γ, which is the union of all families γ_a, $a \in M$. Then the family γ is a uniform covering of the space $(\coprod_{a \in M} X_a, \coprod_{a \in M} U_a)$ and it is a refinement α. Show that γ is point-finite. Let $x \in X$ be an arbitrary point. Let $x \in X_a$, $a \in M$. Since γ_a, $a \in M$ is a point-finite uniform covering of the space (X_a, U_a), $a \in M$, then $x \in X_a$ belongs to only a finite number of elements of the covering γ_a, $a \in M$. Since the spaces (X_a, U_a) and $a \in M$ are disjoint, each point $x \in X$ belongs to only a finite number of elements of the covering γ.
The following theorem shows that strongly uniformly paracompactness is preserved in the preimage direction by uniformly perfect mappings.

Theorem 3

Weakly uniformly paracompactness is preserved in the preimage direction by uniformly perfect mappings.

Proof. Let α be an arbitrary finitely additive open covering of a space (X, U). It is clear that the covering $\{f^{-1}y : y \in Y\}$ refines the covering α. Then $\beta = f^{\#}\alpha = \{f^{\#}A : A \in \alpha\}$, where $f^{\#}A = Y \backslash f(X \backslash A)$, is an open covering of the space (Y, V). Considering all possible finite unions of sets of β, we construct an open covering $\beta^{\angle}$. It is a finitely additive open covering. By the condition of the theorem, there is a point-finite uniform covering $\gamma \in V$ of it. It is easy to see that the covering $f^{-1}\beta^{\angle}$ is a refinement of the covering α. The $f^{-1}\gamma$ is a point-finite uniform covering of the space (X, U) , and it is a refinement of α. So, the uniform space (X, U) is weakly uniformly paracompact.

The following theorem is a uniform analogue of Dowker-Ponomarev-Fedorcuk-Shediva's (Trnkova) theorem for weakly uniformly paracompact space.

Theorem 4

A uniform space (X, U) is weakly uniformly paracompact if and only if for every finitely additive open covering ω of (X, U) there exists a uniformly continuous ω-mapping $f : (X, U) \to (Y, V)$ of the uniform space (X, U) onto a metrizable weakly uniformly paracompact space (Y, V).

Proof. *Necessity*. Let (X, U) be a metrizable weakly uniformly paracompact space and ω be an arbitrary finitely additive open covering. Then the identity map of a space (X, U) is the required uniformly continuous ω-mapping of (X, U) into a metrizable weakly uniformly paracompact space.

Sufficiency. Let ω be an arbitrary finite additive open covering of the space (X, U). Then there exists a uniformly ω-continuous mapping $f : (X, U) \to (Y, V)$ of the uniform space (X, U) onto some metrizable weakly uniformly paracompact space (Y, V). For each point $y \in Y$, there exists a neighborhood O_y whose preimage $f^{-1}O_y$ is contained in some element of the covering ω. Let $\beta = \{O_y : y \in Y\}$. We form an open covering $\beta^{\angle}$ consisting of all possible finite unions of elements of β. We refined a point-finite uniform covering $\gamma \in V$ in it. Then covering $f^{-1}\gamma$ is a refinement of the covering ω of the uniform space (X, U) . We show that $f^{-1}\gamma$ is a point-finite uniformly covering. Indeed, let $x \in X$ be an arbitrary point and $y = f(x)$. Then the point $y \in Y$ belongs to only a finite number of elements of the covering γ. It is easy to see that the point $x \in f^{-1}y$ belongs to only a finite number of elements of the covering $f^{-1}\gamma$.

Therefore, a uniform space (X, U) is weakly uniformly paracompact.

Theorem 5

Any uniformly perfect mapping $f : (X, U) \to (Y, V)$ of a uniform space (X, U) onto a uniform space (Y, V) is an ω-mapping for any finitely additive open covering ω of (X, U).

Proof. Let ω be an arbitrary finitely additive open covering of the space (X, U). It is easy to see that the covering $\alpha = \{f^{-1}y : y \in Y\}$ is a refinement of the covering ω. For each $f^{-1}y \in \alpha$, choose a $W_y \in \omega$ such that $f^{-1}y \subset W_y$. Then from the closedness of the mapping f there exists a neighborhood $O_y \ni y$ such that $f^{-1}O_y \subset W_y$.

Proposition 4

The product of a weakly uniformly paracompact uniform space (X, U) onto a compact uniform space (Y, V) is weakly uniformly paracompact.

Proof. Let (X, U) be a weakly uniformly paracompact space and (Y, V) is a compact uniform space. It is known [see 4, p. 77, Example 1.7.2] that the projection $\pi_X : (X, U) \times (Y, V) \to (X, U)$ is uniformly perfect. Then it is an ω-mapping of the product $(X, U) \times (Y, V)$ onto a weakly uniformly paracompact space (Y, V) for any finitely additive open covering ω of $(X, U) \times (Y, V)$. Therefore, according to Theorem 4, the uniform space $(X, U) \times (Y, V)$ is weakly uniformly paracompact.

Any uniformly A-paracompact space is weakly uniformly paracompact. The converse is generally not true.
The following theorem is an intrinsic characteristic for strongly uniformly paracompact spaces.

Theorem 6

For a uniform space (X, U) the following are equivalent :

1. (X, U) is uniformly A-paracompact ;
2. (X, U) is weakly uniformly paracompact and the topological space (X, τ_U) is paracompact.

Proof. 1) $\Rightarrow$ 2). It is obviously.
2) $\Rightarrow$ 1). Let α be an arbitrary finitely additive open covering of the uniform space (X, U). We refined a locally finite open covering β in it. We form a covering $\beta^{\angle}$ consisting of all possible finite unions of elements of β. Then $\beta^{\angle}$ is a finitely additive open locally finite covering. Next, the covering $\beta^{\angle}$ we has a refinement point-finite which is a finite uniform covering $\gamma \in U$. Therefore, a locally finite uniform covering $\beta^{\angle}$ is a refinement of the finite additive open covering α. Thus, the uniform space (X, U) is uniformly A-paracompact.

References

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